

Anyons, Monopole and Coulomb Problem

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Abstract

The monopole systems with hidden symmetry of the two-dimensional Coulomb problem are considered.

One of them, the “charge-charged magnetic vortex” (“charge- Z_2 -dyon”) with a half-spin, is constructed by reducing the quantum circular oscillator with respect to the action of the parity operator.

The other two systems are constructed by reduction from the two-dimensional complex space. The first system is a particle on the sphere in the presence of the exterior constant magnetic field (generated by Dirac’s monopole located in its center). This system is dual to the massless (3+1)-dimensional particle with fixed energy. The second system represents the particle on the pseudosphere in the presence of exterior magnetic field and is dual to the massive relativistic anyon.

1 Introduction

As it is well-known, most of the integrable systems of classical mechanics can be constructed by the Hamiltonian reduction of the simplest systems of the type of a free particle or an oscillator formulated on a larger phase space [1]. This kind of construction not only classifies the known integrable systems but also allows one to construct new integrable systems together with their explicit solutions.

The procedure of reduction becomes especially important in quantum mechanics where it is much more difficult to establish integrability and exact solvability of systems. For example, though any one-dimensional system of classical mechanics is integrable and exactly solvable, the corresponding quantum-mechanical system being integrable is exactly solvable only in exceptional cases.

On the other hand, the Hamiltonian reduction provides a natural framework to construct integrable *spinning* systems, including multiparticle integrable ones (see, e.g., [2]).

An interesting feature is that such two-, three- and higher dimensional systems can be interpreted as the ones interacted with external gauge fields of monopoles. Correspondingly, the spin in these systems can be viewed as a consequence of the

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nontrivial topology induced by the presence of a monopole. Moreover, in the dual interpretation, i.e. under exchange of the role of coordinates and momenta, these systems can be interpreted as free relativistic systems of spinning particles in an appropriate gravitational background.

A good illustration is a Hamiltonian reduction of the four-dimensional isotropic oscillator under the $U(1)$ -group action with the nonzero value s of the Hamiltonian generator of that group.

As a result of that reduction, the oscillator phase space $T_*\mathbb{C}^2$ reduces to the phase space of a nonrelativistic charge-Dirac monopole system which is $T_*\mathbb{R}^3$ with the symplectic structure

$$dp_a \wedge dq^a + s\varepsilon_{abc} \frac{q^a}{|q|^3} dq^b \wedge dq^c, \quad a, b, c = 1, 2, 3 \quad (1.1)$$

and the rotation generators

$$J^a = \varepsilon^{abc} p_b q_c + s \frac{q^a}{|q|}. \quad (1.2)$$

The energy levels of the oscillator reduce to the negative energy levels of the non-relativistic “charge-dyon” system described by the Hamiltonian [3]

$$H = \frac{p^2}{2\mu} - \frac{s^2}{2\mu|q|^2} + \frac{\alpha}{|q|} \quad (1.3)$$

Notice that the “centrifugal” term in this Hamiltonian provides the interaction of the induced dipole momentum of the systems with the monopole magnetic field [4].

In the dual picture, i.e. after the exchange of the role of the coordinates and momenta, the reduced phase space describes (3+1)-dimensional massless particles with the helicity s . This duality becomes clear in a four-dimensional picture where the prototypes of phase space variables and the generators of spatial rotations define the well-known twistor realization of the (3+1)-dimensional Poincaré group for massless particles.

When the initial system reduces with respect to Hamiltonian action of the non-Abelian group, the resulting system acquires isospin degrees of freedom. An example is the generalization of the five-dimensional Coulomb problem, which includes the interaction with the $SU(2)$ monopole field. This system was constructed by reducing the eight-dimensional isotropic oscillator under the $SU(2)$ -group action both in the classical [5] and quantum [6] cases.

In the general case, the system obtained by reduction under Hamiltonian action of the G group, possesses an interaction with an external G -field, and the values of the generators of the G -group play the role of charges [7].

In the present paper, we consider the above mechanisms on the systems which are related to the two-dimensional Coulomb problem.

In *Section 2*, we show that despite the classical equivalence of the two-dimensional Coulomb problem and circular oscillator, established by the Bohlin-Levi-Chivita transformation, the quantum correspondence between these systems requires reduction of the quantum circular oscillator by the parity operator (Z_2 -group) action. As a result, the oscillator breaks into two systems, the two-dimensional Coulomb problem, and its generalization, with half-spin induced by the magnetic vortex (or, equivalently, two-dimensional “charge- Z_2 -dyon” system) [9].

In *Section 3*, we perform the Hamiltonian reduction of four-dimensional systems to particles on the sphere and pseudosphere in homogeneous magnetic fields.

The first system is a particle on a sphere which interacts with a magnetic monopole located in its center, and it is dual to the (3+1)-dimensional massless particle with fixed energy. The second system has a dual interpretation as a free massive relativistic anyon [10].

2 Coulomb Problem with Magnetic Vortex

Consider the correspondence between the two-dimensional Coulomb problem and classical two-dimensional isotropic oscillator (circular oscillator) [8].

It is convenient to describe the Hamiltonian system, corresponding to the circular oscillator, in terms of the phase space $T_*\mathbb{C}$ by using the following canonical Poisson brackets and Hamiltonian

$$\{\pi, z\} = 1, \quad \{\bar{\pi}, \bar{z}\} = 1; \quad H_{osc} = \omega(\pi\bar{\pi} + z\bar{z}). \quad (2.1)$$

This system has three integrals of motion which forms the $su(2)$ algebra: the oscillators rotational momentum J and the two other generators which are convenient to be presented in the form of the complex (vector) integral of motion I^+ and its complex-conjugated one I^- :

$$J = i(\pi z - \bar{\pi} \bar{z}), \quad I^+ = \omega(\pi^2 - \bar{z}^2) : \quad \{I^+, I^-\} = 2\omega^2 J, \quad \{I^\pm, J\} = \pm 2I^\pm \quad (2.2)$$

Let us exclude the origin of the coordinates of \mathbb{C} , and perform the transformation $(z, \pi) \rightarrow (w, p)$, which is canonical on the space $T_*\dot{\mathbb{C}}$ (and singular in the origin of the coordinates of \mathbb{C}):

$$w = z^2, p = \pi/2z : \quad \{w, w\} = 0, \quad \{w, p\} = 1, \quad \{p, \bar{p}\} = 0. \quad (2.3)$$

This transformation, after rescaling $w \rightarrow w\sqrt{2\mu\omega}$, $p \rightarrow p/\sqrt{2\mu\omega}$, converts the energy levels of the oscillator $H_{osc} = E_{osc}$ into those of the Coulomb problem $H_C = -2\mu\omega^2$ where

$$H_C = \frac{1}{2\mu} p\bar{p} - \frac{E_{osc}}{|w|}; \quad (2.4)$$

The oscillator integrals of motion are transform into the rotational momentum and Runge-Lenz vector of the two-dimensional Coulomb problem

$$2\tilde{J} = 2i(wp - \bar{w}\bar{p}), \quad \tilde{I} = \frac{i(Jp + pJ)}{2} - \frac{E_{osc}w}{4|w|}. \quad (2.5)$$

Note that in passing from the oscillator to the Coulomb problem the rotational momentum of the system is doubled. This reflects the fact, that the single round of the orbit in the oscillator problem corresponds to the double round of the orbit in the Coulomb one. To verify this, we parametrize the oscillator trajectories by Zhukovski's ellipse $z = u + 1/u$ where the complex parameter u parametrizes a circle with a radius different from unity, $|u| = \text{const} \neq 1$. Then, as a result of transformation $w = z^2$ we have

$$z = u + 1/u \rightarrow w = u^2 + 1/u^2 + 2,$$

which means that the center of attraction of the Coulomb problem is shifted to the focus of ellipse, and a single round of the ellipses by the particle in the oscillator problem corresponds to the double rounding of the orbit in the Coulomb one.

So, when we deal with elliptic orbits, we can ascertain the equivalence of the circular oscillator and two-dimensional Coulomb problems on the classical level.

In the corresponding quantum problem we have

$$H_{osc}(\hat{\pi}, \hat{\pi}, z, \bar{z})\Psi = E_{osc}\Psi, \quad \Psi(|z|, \arg z) = \Psi(|z|, \arg z + 2\pi), \quad (2.6)$$

where $\hat{\pi} = -i\hbar\partial_z$, $\hat{\pi} = -i\hbar\partial_{\bar{z}}$.

Though transformation (2.3) turns the Schrödinger equation of the oscillator into one for the two-dimensional Coulomb problem, it breaks the single-valuedness requirement for the wave functions, $\Psi(|w|, \arg w + 4\pi) = \Psi(|w|, \arg w)$, i.e. the quantum circular oscillator problem transform into the Coulomb problem on the *two-sheet Riemann surface*.

Therefore, we should supply the transformation (2.3) with the reduction of the oscillator Schrödinger equation by the Z_2 -group action given by the parity operator. This means that we may restrict ourselves to the even ($\sigma = 0$) or odd ($\sigma = \frac{1}{2}$) solutions of the oscillator Schrödinger equation

$$\Psi_\sigma(z, \bar{z}) = \psi_\sigma(z^2, \bar{z}^2)e^{i\sigma\arg z}. \quad (2.7)$$

Then, the wave functions ψ_σ obey the condition

$$\psi_\sigma(|w|, \arg w + 2\pi) = \psi_\sigma(|w|, \arg w);$$

therefore, the domain of definition $\arg w \in [0, 4\pi)$ can be reduced, without loss of generality, to $\arg w \in [0, 2\pi)$.

This procedure results in the Schrödinger equation

$$\left(\frac{1}{2\mu}\hat{p}_\sigma\hat{p}_\sigma^+ - \frac{\alpha}{|w|} + \mathcal{E}\right)\psi_\sigma = 0, \quad (2.8)$$

where $\mathcal{E} = \mu\omega^2/8$, $\alpha = E/4$, and the momentum operator is defined by the expression

$$\hat{p}_\sigma = -2i\hbar\partial_w - \frac{i\hbar\sigma}{2w}. \quad (2.9)$$

Substitution of (2.9) into (2.5) gives us the expressions for the integrals of motion of the reduced system.

So, for both the values of σ , the obtained system is characterized by the symmetry of the two dimensional Coulomb problem.

From the eigenvalues of the oscillator rotational momentum M and energy E

$$E = \hbar\omega(2N_r + |M| + 1), \quad M = 0, \pm 1, \pm 2, \dots \quad N_r = 0, 1, 2, \dots, \quad (2.10)$$

we immediately derive the eigenvalues for the rotational momentum m_σ and for the energy \mathcal{E}^s of the reduced system

$$\mathcal{E}^s = -\frac{\mu\alpha^2}{\hbar^2(N_r + |m_\sigma| + 1/2)^2}, \quad m_\sigma = \pm\sigma, \pm(1 + \sigma), \pm(2 + \sigma), \dots \quad (2.11)$$

The wave functions of the reduced system can easily be obtained from the oscillator ones as well.

The Schrödinger equation of the reduced system describes the motion of spinless particle with the electric charge e in the electric and magnetic fields with the potentials

$$\phi = -\frac{\alpha}{e|w|}, \quad A_w = \sigma\frac{i\hbar c}{2ew}, \quad (2.12)$$

i.e. in the field of Z_2 -dyon. Indeed, the magnetic potential A_w defines the magnetic field with zero strength $B = \text{rot}A_w = 0$ ($w \in \dot{\mathbb{C}}$) and constant magnetic flux $\sigma\pi\hbar c/2e$.

In other words, in the case of $\sigma = 0$ we have the two-dimensional hydrogen atom, whereas for $\sigma = \frac{1}{2}$ we have the “charge-charged magnetic vortex” (“charge- Z_2 -monopole”) system.

As we see from (2.11), the second system possesses spin 1/2 and the Aharonov-Bohm effect is observed in it. Note that the Hamiltonian of the system (2.8) can be represented in the form analogous to the one for three-dimensional “charge-Dirac dyon” system (1.3)

$$\hat{H}_\sigma = -\frac{4\hbar^2}{2\mu}\partial_w\bar{\partial}_w + \frac{\hbar^2\sigma^2}{2\mu|w|^2} - \frac{\alpha}{|w|}.$$

In a similar way, we can consider the transformation $w = z^N$ corresponding to the reduction by Z_N -group. As a result of that reduction, the initial two-dimensional system with the potential $|z|^a$ splits into N “charge-magnetic vortex” (or “charge- Z_N -monopole”) bound systems with the spin $\sigma = 0, 1/N, 2/N, \dots, (N-1)/N$ and potential $|w|^b$, where $(a+2)(b+2) = 4$, $N = 1 + a/2$.

3 Particle on Sphere and Pseudosphere

The hidden symmetry of the Coulomb problem establishes its trajectorial correspondence with an isotropic oscillator only in the two-, three-, and five-dimensional cases.

In the general case, the n -dimensional Coulomb problem is trajectory-equivalent to a geodesic flux on the n -dimensional sphere (attractive problem) or n -dimensional pseudosphere (repulsive problem). Integrals of motion of the Coulomb problem correspond to the rotation generators on the (pseudo)sphere. Equivalence between these systems in the two-dimensional case can be established as follows:

$$(1 + pp^+)^2 ww^+ - mE = 0 \Leftrightarrow pp^+ - \frac{\sqrt{mE}}{\sqrt{ww^+}} + 1 = 0$$

It can be easily verified that the Hamiltonian of a particle on a sphere (pseudosphere) under transformation (2.3) goes over into the squared Hamiltonian of a circular oscillator with the positive (negative) coupling constant

$$(1 \pm p\bar{p})^2 w\bar{w} = (\pi\bar{\pi} \pm z\bar{z})^2.$$

Hereafter, we take advantage of the stereographic projection of a sphere, \mathbb{CP}^1 and Poincaré model for pseudosphere (Lobachevsky plane) \mathcal{L} , where the metric is defined by the expression

$$g(p, p^+) = \frac{mdpdp^+}{(1 + pp^+)^2}, \quad (3.1)$$

where $p^+ = \bar{p}$ for \mathbb{CP}^1 and $p^+ = -\bar{p}$ for \mathcal{L} .

Notice that \mathbb{CP}^1 and \mathcal{L} can be constructed by the Hamiltonian reduction of \mathbb{C}^2 and $\mathbb{C}^{1,1}$ equipped by the Kähler structures $d\pi^\alpha d\bar{\pi}_\alpha$ under the $U(1)$ -group action which is defined by the generator $P = \pi^\alpha \bar{\pi}_\alpha$ with the value $P = m$. Hereafter, indices are raised and lowered by using the metric $\eta_{\alpha\bar{\beta}}$ where $\eta_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}}$ on \mathbb{C}^2 and $\eta_{\alpha\bar{\beta}} = \sigma_{\alpha\bar{\beta}}^3$ on $\mathbb{C}^{1,1}$.

The canonical Poisson brackets on \mathbb{C}^2 and $\mathbb{C}^{1,1}$ are reduced to those corresponding to the Kähler structures (3.1) under the following choice of coordinates of the reduced phase spaces $p_{(0)} = \pi^1/\pi^0$, $\bar{p}_{(0)} = \bar{\pi}_1/\bar{\pi}_0$ in the domain $\pi^0 \neq 0$, and $p_{(1)} = \pi^0/\pi^1$, $\bar{p}_{(1)} = \bar{\pi}_0/\bar{\pi}_1$, in the domain $\pi^1 \neq 0$. So, a sphere can be covered with two charts with the transition functions $p_{(0)} = 1/p_{(1)}$, where p parametrizes the complex plane, while a pseudosphere can be covered with only one chart parametrized by the coordinate $p_{(0)}$, $|p| < 1$ at $m < 0$ ($p_{(1)}$, $|p| > 1$, at $m > 0$).

Now consider the phase spaces $T_*\mathbb{C}^2$ and $T_*\mathbb{C}^{1,1}$, equipped with the canonical Poisson brackets

$$\{\pi^\alpha, \omega_\beta\} = \delta_\beta^\alpha, \quad \{\bar{\pi}^{\bar{\alpha}}, \bar{\omega}_{\bar{\beta}}\} = \delta_{\bar{\beta}}^{\bar{\alpha}}, \quad \alpha, \beta = 0, 1. \quad (3.2)$$

Let us carry out the Hamiltonian reduction of these spaces by the action of the generators

$$P = \pi^\alpha \bar{\pi}_\alpha, \quad J = \frac{i}{2}(\pi^\alpha \omega_\beta - \bar{\omega}^{\bar{\alpha}} \bar{\pi}_{\bar{\alpha}}), \quad \{P, J\} = 0, \quad (3.3)$$

with the values

$$J = s, P = m. \quad (3.4)$$

Due to commutativity of these generators, reduced phase spaces are four-dimensional. The following functions (p, w) can be chosen as local complex coordinates of the reduced phase space (when $\pi^0 \neq 0$)

$$p = \pi^1/\pi^0, \quad w = ig(p, \bar{p})(\lambda^1 - p\lambda^0), \quad \lambda^\alpha = \omega^\alpha/\bar{\pi}^0, \quad (3.5)$$

where the metric $g(p, \bar{p})$ is defined by expression (3.1).

The reduced Poisson brackets on these spaces are of the form

$$\{p, w\} = \{\bar{p}, \bar{w}\} = 1, \quad \{w, \bar{w}\} = i\frac{s}{m}g(p, \bar{p}). \quad (3.6)$$

Thus, we have reduced the canonical Poisson brackets on $T_*\mathbb{C}^2$ and $T_*\mathbb{C}^{1,1}$ to the twisted ones on $T_*\mathbb{CP}^1$ and \mathcal{L} , respectively.

We have got the phase spaces for the particle on the sphere and pseudosphere in the presence of the constant homogeneous magnetic field.

For the sphere this field is generated by the magnetic monopole located in the center of the sphere.

Now let us define on $T_*\mathbb{C}^2$ and $T_*\mathbb{C}^{1,1}$ the generators

$$P^a = \pi^\alpha T_{\alpha\bar{\beta}}^a \bar{\pi}^{\bar{\beta}}, \quad J^a = \frac{1}{2}T_{\alpha\bar{\beta}}^a(\pi^\alpha \bar{\omega}^{\bar{\beta}} + \omega^\alpha \bar{\pi}^{\bar{\beta}}), \quad (3.7)$$

which form the algebra

$$\{P^a, P^b\} = 0, \quad \{P^a, J^b\} = \varepsilon^{abc}P_c, \quad \{J^a, J^b\} = \varepsilon^{abc}J_c, \quad (3.8)$$

where $T^a = \sigma^a$ for \mathbb{C}^2 and $T^a = (\sigma^0, \sigma^1, \sigma^2)$ for $\mathbb{C}^{1,1}$; the vector indices a, b, c are raised and lowered by using the metric $g_{ab} = \frac{1}{2}\text{tr}(T^a \eta T^b \eta)$.

Correspondingly, for \mathbb{C}^2 we have $g_{ab} = \delta_{ab}$, and (3.8) is the $e(3)$ algebra while for $\mathbb{C}^{1,1}$ we have $g_{ab} = \text{diag}(1, -1, -1)$, and (3.8) is the $(2+1)$ - dimensional Poincaré algebra $iso(1,2)$.

The generators (3.7) commute with P and J , and therefore, can be reduced on $T_*\mathbb{CP}^1$ and \mathcal{L} where they take the form

$$J^a = V^a(p)w + \bar{V}^a(\bar{p})\bar{w} + \frac{s}{m}P^a, \quad P^a = m \frac{T_{00}^a + T_{10}^a \bar{p} + T_{01}^a p + T_{11}^a p \bar{p}}{1 + p \bar{p}}. \quad (3.9)$$

with $V^a(p) = i\partial_p P^a(p, \bar{p})$.

So, the system with such a phase space possesses the own angular momentum (“spin”) $s^a = \frac{s}{m} P^a$.

The Hamiltonian of the free particle on the sphere (pseudosphere) is related with J^a by the expression

$$H = g^{-1} w \bar{w} = J^a J_a - s^2, \quad (3.10)$$

i.e. can be considered as a reduced spherical top.

Performing the Hamiltonian reduction of the initial phase space only by the action of J , we get the six-dimensional phase space. Thus, choosing the coordinates of the reduced phase space of the form

$$P^a = \pi T^a \bar{\pi}, \quad Q^a = i(\omega T^a \pi - \pi T^a \omega)/(2\pi \bar{\pi}),$$

we see that $T_* \mathbb{C}^2$ ($T_* \mathbb{C}^{1,1}$) is reduced to $T_* \mathbb{R}^3$ ($T_* \mathbb{R}^{1,2}$) while the reduced symplectic structure and the reduced generators J^a take the form (1.1) and (1.2) after canonical transformation $(Q^a, P^a) \rightarrow (P^a, -Q^a)$.

It is easy to see that $P^\mu = (P, P^a)$, $M^{[\mu\nu]} = (J^a, (2\pi \bar{\pi}) Q^a)$ define on $T_* \mathbb{C}^2$ the well-known twistor realization of the (3+1)-dimensional Poincaré group for the massless particles, and J plays the role of their helicity. This gives transparent explanation of the duality between massless particles and nonrelativistic “charge-monopole” system mentioned in the Introduction.

One can interpret the systems, reduced to sphere, in term of the (3+1)-dimensional Poincaré group. In this case, the radius of the sphere has a meaning of the energy of the massless particle whereas the spin of the system has a meaning of the helicity.

However, since the generator P does not commute with the generators of the Lorentz rotation, the performed Hamiltonian reduction to the sphere destroys the Lorentz invariance of the reduced system.

Unlike the system on the sphere, the one on the pseudosphere admits complete relativistic interpretation [10].

Namely, the generators P^a, J^a define on $T_* \mathbb{C}^{1,1}$ the action of the (2+1)-dimensional Poincaré group and commute with J and P . Thus, the reduced system is relativistic-invariant.

The “Casimirs” $P^a P_a$ and $P_a J^a$ satisfy the following conditions:

$$P^a P_a = P^2, \quad P_a J^a = P J.$$

Hence, the generators J and P can be interpreted as those of spin and mass of the relativistic particle. Note that the transformation $\pi^1 \leftrightarrow \pi^0, \omega^0 \rightarrow \omega^1$, leads to $(m, s) \rightarrow (-m, -s)$ and, consequently, is an identity transformation in terms of the Poincaré group. So, the Hamiltonian reduction by the action of the generators P and J fixes the mass $|m|$ and spin s of a free relativistic two-dimensional particle.

Let us now outline the scheme of an analogous reduction in the quantum case. Introduce the operators

$$\hat{\omega}_\alpha = \partial/\partial\pi^\alpha, \quad \hat{\omega}^\alpha = -\partial/\partial\bar{\pi}_\alpha,$$

and impose on the wave function $\Psi(\pi, \bar{\pi})$ the conditions

$$\hat{P}\Psi(\pi, \bar{\pi}) = m\Psi(\pi, \bar{\pi}), \quad \hat{J}\Psi(\pi, \bar{\pi}) = s\Psi(\pi, \bar{\pi}) \quad (3.11)$$

which are the quantum analogs of the constraints (3.4).

The solution to this system can be represented in the form

$$\Psi(\pi, \bar{\pi}) = \psi_{m,s}(p, \bar{p})e^{is\gamma(\pi, \bar{\pi})}, \quad \text{where} \quad \hat{J}\gamma = i, \quad (3.12)$$

from which it follows:

$$\hat{w}\Psi = g(p, \bar{p})(\hat{\lambda}^1 - p\hat{\lambda}^0)\Psi = e^{is\gamma}\hat{w}_{red}\psi(p, \bar{p}), \quad (3.13)$$

where

$$\hat{w}_{red} = i\frac{\partial}{\partial p} + \frac{s}{2m}A(p, \bar{p}), \quad A_\gamma(p, \bar{p}) = \hat{w}\gamma. \quad (3.14)$$

So, $\psi(p, \bar{p})$ and \hat{w}_{red} define the wave function and the momentum operator of the reduced system.

Choosing $\gamma_+ = i\log \pi^0/\bar{\pi}_0$ or $\gamma_- = i\log \pi^1/\bar{\pi}_1$, we find $A_+(p, \bar{p}) = i\partial_p K(p, \bar{p})$ and $A_-(p, \bar{p}) = i\partial_{\bar{p}} K(p, \bar{p})$, respectively, where $K(p, \bar{p}) = m\log(1 + p\bar{p})$.

We can also choose $\gamma = (\gamma_+ + \gamma_-)/2$. In this case, each chart contains a singularity in $p = 0$, while the choice γ_\pm at $p = p_\pm$ leads to a system, being regular on each chart.

The reduction of the initial system by only generator J suggest the ansatz $\Psi(\pi, \bar{\pi}) = \psi(P^a)e^{is\gamma}$ which results in the mometum operator with the vector potential of the Dirac monopole at $\gamma = \gamma_\pm$, and vector potential of the Schwinger one at $\gamma = (\gamma_+ + \gamma_-)/2$.

It is clear that $\psi_{m,s}(p, \bar{p}) = \psi_s(P^a)$ where P^a are defined by (3.9).

The choice of γ depends on a given problem. For example, the requirement of invariance of the wave function on the sphere under transition from one chart to another implies the choice $\gamma = (\gamma_+ + \gamma_-)/2$ which corresponds to the Schwinger monopole. If $\gamma = \gamma_\pm$, the reduced system is regular on both the charts, i.e. we have Dirac's monopole.

The requirement for the wave function to be single-valued leads to the quantization of spin: it takes integer or half-integer values.

Now, consider the system on $\mathbb{C}^{1,1}$ corresponding to that on $\mathbb{R}^{1,2}$ and on the pseudosphere.

In this case the choice $\gamma = \gamma_\pm$ leads to systems with regular vector potentials on both $\mathbb{R}^{1,2}$ and the pseudosphere (since the later can be covered with one chart).

Therefore, the requirement of regularity does not impose any quantization condition on s . As we have shown, a system on a pseudosphere can be interpreted as a two-dimensional free relativistic particle, that is an anyon, because its spin is not quantized[10],[11].

One can look at this picture from another point of view. We have shown that $(\pi^1, \omega^1) \leftrightarrow (\pi^0, \omega^0)$ is the identity transformation for a two-dimensional relativistic particle. Taking $\gamma = (\gamma_+ + \gamma_-)/2$, we obtain that the requirement of the single-valuedness of the wave function Ψ_s does not lead to the quantization of s due to the signature of the metric on $\mathbb{C}^{1,1}$. On the other hand, the initial system with a given choice of γ can be viewed as the system of two identical particles. Thus, interchanging these particles, we find that the wave function acquires the phase $2\pi s$.

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